Lecture 28

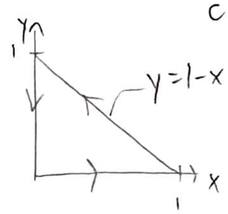
Recall, $\int_a^b F'(x)\,dx = F(b) - F(a)$ relates the derivative of function to the value of the function on the boundary of the domain. This can be extended to two dimensions.

Green's Theorem

Let R be a simple region in the xy-plane with a piecewise smooth boundary, C, oriented counterclockwise. Also, let M & N be functions of two variables with continuous partial derivatives on R.
Then,

$$\int_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Ex. 1 Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = \langle x^4, xy \rangle$ and C is the trianglur curve



$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} y - \emptyset \,dy\,dx$$

$$= \int_0^1 \frac{1}{2}y^2 \Big|_0^{y=1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= -\frac{1}{6}(1-x)^3\Big|_0^1 = \frac{1}{6}$$

Ex. 2 Evaluate $\oint_C (3y - e^{\sin(x)})\,dx + (7x + \sqrt{y^4+1})\,dy$ where C is the circle $x^2+y^2=9$ oriented CW.

Apply Green's thm

$$-\iint_D \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin(x)})\right] dA$$

b/c it's CW

$$= -\iint_D (7-3)\, dA$$

Let's use polar coordinates

$$-4\int_0^{2\pi}\int_0^3 r\, dr\, d\theta = -36\pi$$

<u>Area</u>

Recall, $A = \iint_R 1\, dA$. If R is simple, this means we can choose M + N s.t. $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$ so that

$$\iint_R 1\, dA = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_C M(x,y)\,dx + N(x,y)\,dy$$

We typically pick one set of these three:

1) $M = 0$, $N = x$

2) $M = -y$, $N = 0$

3) $M = -\frac{1}{2}y$, $N = \frac{1}{2}x$.

These yield,

$$A = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int x\,dy - y\,dx$$

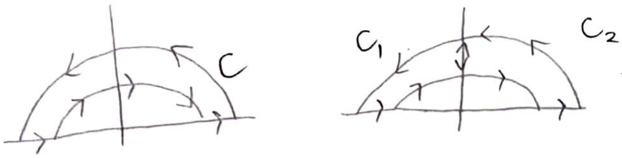
<u>Ex. 3</u> Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

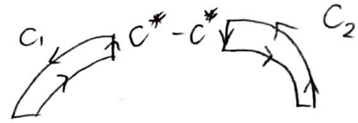
$\bar{r}(t) = \langle a\cos(t), b\sin(t)\rangle$ for $0 \le t \le 2\pi$

$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a\cos(t)\, b\cos(t) - b\sin(t)(-a\sin(t))\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\, dt = \pi ab$$

Application to Non-Simple Regions





$$C = C_1 + C_2 + C^* - C^*$$

$$C = C_1 + C_2$$

$$\int_C M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$$